Probabilistic Methods in Combinatorics

Instructor: Oliver Janzer

Solutions to Assignment 5

Problem 1. Prove that there is an absolute constant $c > 0$ with the following property. Let A be an n by n matrix with pairwise distinct entries. Then there is a permutation of the rows of A so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

Solution. Let c be a constant which we will specify later and let $k = \lceil c\sqrt{n} \rceil$. Take a permutation of the rows of A uniformly at random. Let the permuted matrix be B. For any $1 \le j \le n$ and any $1 \le i_1 < i_2 < \cdots < i_k \le n$, let $T(j; i_1, \dots, i_k)$ be the event that the sequence $B_{i_1,j}, B_{i_2,j}, \dots, B_{i_k,j}$ is increasing. Note that B contains an increasing subsequence of length at least $c\sqrt{n}$ in some column if and only if at least one of the events $T(j; i_1, \dots, i_k)$ occurs.

However, the probability of each $T(j; i_1, \dots, i_k)$ is $1/k!$ since any ordering of the elements $B_{i_1,j}, \dots, B_{i_k,j}$ is equally likely. Hence, by the union bound,

$$\mathbb{P}(B \text{ contains an increasing subsequence of length at least } c\sqrt{n} \text{ in some column}) \le n\binom{n}{k} \cdot \frac{1}{k!}.$$

Using the approximations $\binom{n}{k} \le (\frac{ne}{k})^k$, $k! \ge \sqrt{2\pi k} \cdot k^k e^{-k}$ and $k \ge c\sqrt{n}$, we get

$$n\binom{n}{k} \cdot \frac{1}{k!} \le \frac{n}{\sqrt{2\pi k}} \left(\frac{ne^2}{k^2}\right)^k \le \frac{n}{\sqrt{2\pi k}} \cdot \left(\frac{e^2}{c^2}\right)^k \le \frac{n^{3/4}}{\sqrt{2\pi c}} \left(\frac{e^2}{c^2}\right)^{c\sqrt{n}}.$$

If $c > e$, then this expression tends to 0 as $n \to \infty$. So we can choose a constant c for which

$$\mathbb{P}(B \text{ contains an increasing subsequence of length at least } c\sqrt{n} \text{ in some column}) < 1$$

for all n. This means that for this choice of c, any matrix A has a permutation of its rows so that no column in the permuted matrix contains an increasing subsequence of length at least $c\sqrt{n}$.

Problem 2. Prove that every three-uniform hypergraph with n vertices and $m \ge n/3$ hyperedges contains an independent set (i.e. a set of vertices containing no hyperedges) of size at least

$$\frac{2n^{3/2}}{3\sqrt{3}\sqrt{m}}.$$

Let S be a random subset of vertices where every vertex is chosen independently with probability $p = \sqrt{n/(3m)}$. Note that by assumption, $0 \le p \le 1$, so p is indeed a valid probability. For every edge whose all three vertices are in S, remove at most one vertex. Thus we obtain an independent set I. Let $X = |S|$ and Y be the number of edges whose all three vertices are in S. Then $|I| \ge X - Y$. Showing $\mathbb{E}[X-Y] \ge \frac{2n^{3/2}}{3\sqrt{3m}}$, finishes the proof by the first moment method. Using linearity of expectation, we have $\mathbb{E}[X] = np$ and $\mathbb{E}[Y] = mp^3$. Plugging in the value of p, we get $\mathbb{E}[X-Y] = \frac{2n^{3/2}}{3\sqrt{3m}}$, as claimed.

Problem 3. Prove that if there exists some $0 \le p \le 1$ such that

$$\binom{n}{t}p^{\binom{t}{2}} + \binom{n}{k}(1-p)^{\binom{k}{2}} \le n/2,$$

then $R(t,k) \ge n/2$. Using this, show that the Ramsey number $R(4,k)$ satisfies

$$R(4,k) \ge \Omega((k/\ln k)^2).$$

Solution. Define a random red-blue colouring of the edges of K_n as follows. Colour every edge red with probability p and blue with probability $1-p$. Then the expected number of red cliques of size t is $\binom{n}{t}p^{\binom{t}{2}}$, while the expected number of blue cliques of size k is $\binom{n}{k}(1-p)^{\binom{k}{2}}$. Hence, there exists a colouring in which the number of blue cliques of size t plus the number of red cliques of size k is at most $\binom{n}{t}p^{\binom{t}{2}} + \binom{n}{k}(1-p)^{\binom{k}{2}}$. By assumption, this is at most $n/2$. Hence, we can delete at most $n/2$ vertices and get rid of all red cliques of size t and blue cliques of size k. The remaining graph has at least $n/2$ vertices and it has no forbidden clique, so $R(t,k) \ge n/2$.

We will now use this result with $t = 4$ to prove $R(4,k) \ge \Omega((k/\ln k)^2)$. Take $n = c(k/\ln k)^2$ for a sufficiently small positive constant c. We need to prove the existence of $0 \le p \le 1$ which satisfies $\binom{n}{4}p^6 + \binom{n}{k}(1-p)^{\binom{k}{2}} \le n/2$. To make sure that the first summand is at most $n/4$, we take $p = n^{-1/2} = c^{-1/2} \cdot \frac{\ln k}{k}$. Then

$$\binom{n}{k}(1-p)^{\binom{k}{2}} \le n^k e^{-p\binom{k}{2}} = (ne^{-p\frac{k-1}{2}})^k.$$

Note that if c is a sufficiently small positive constant, then $ne^{-p\frac{k-1}{2}} = c(k/\ln k)^2 e^{-c^{-1/2}\cdot\frac{\ln k}{k}\cdot\frac{k-1}{2}} < 1$, so in this case $\binom{n}{4}p^6 + \binom{n}{k}(1-p)^{\binom{k}{2}} \le n/4 + 1 < n/2$.